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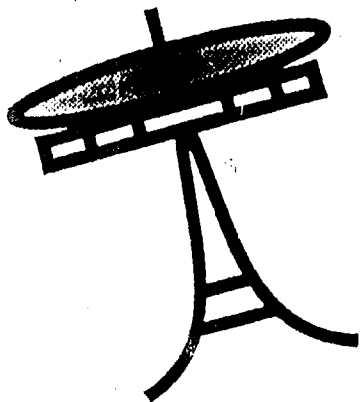
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1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE August 1991		3. REPORT TYPE AND DATES COVERED professional paper	
4. TITLE AND SUBTITLE EXTENSION OF THE MEASURE-FREE APPROACH TO CONDITIONING OF FUZZY SETS AND OTHER LOGICS				5. FUNDING NUMBERS PR: ZT52 PE: 0601152N WU: DN306225	
6. AUTHOR(S) I. R. Goodman and D. W. Stein				8. PERFORMING ORGANIZATION REPORT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Ocean Systems Center San Diego, CA 92152-5000					
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Office of Chief of Naval Research Independent Research Programs (IR) OCNR-10P Arlington, VA 22217-5000				10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES					
12a. DISTRIBUTION/AVAILABILITY STATEMENT  Approved for public release; distribution is unlimited.				12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words)  Drs. Goodman and Nguyen define conditioning in a probabilistic setting. The conditional event algebra, which they associate to a Boolean algebra, allows one to consider conditional probabilities as events, and not just as measures. This facilitates combining conditional events with different antecedents, which is important in solving data fusion problems. In the present work we develop conditioning in a more general algebraic context which includes Boolean algebras and Zadeh's fuzzy set theory. Our structure, roughly speaking, has the properties of a Boolean algebra except for the law of excluded middle. Using functional image extension, the principle used to define the operations, we define disjunction, conjunction, and complement operations on the conditional objects. Unlike the probability case, closure problems arise. They are handled by extending the set of conditional objects. We define a semantic evaluation on the extended set which, when applied to Boolean algebras, yields the familiar conditional probability evaluations and which associates to each conditional fuzzy set a fuzzy set membership function. L. Zadeh, E. Hisdall, and H. Nguyen have also studied conditional fuzzy set membership functions.  Published in <i>Proceedings Third International Fuzzy Systems Association</i> , 1989.					
14. SUBJECT TERMS  combination of evidence data fusion game theory  uncertainty measures				15. NUMBER OF PAGES	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED				18. PRICE CODE	
18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED		19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED		20. LIMITATION OF ABSTRACT SAME AS REPORT	

01 9 12 178

UNCLASSIFIED

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**3rd IFSA congress**

**University of Washington  
Seattle, Washington, USA  
August 6-11, 1989**

**IFSA : The International Fuzzy Systems Association**



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NTIS GRAB	<input checked="" type="checkbox"/>
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Unannounced	<input type="checkbox"/>
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Dist	Avail and/or Special
A-1	20

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# Extension of the Measure-Free Approach to Conditioning of Fuzzy Sets and Other Logics

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## Abstract

Drs. Goodman and Nguyen define conditioning in a probabilistic setting [2]–[5]. The conditional event algebra, which they associate to a Boolean algebra, allows one to consider conditional probabilities as events, and not just as measures. This facilitates combining conditional events with different antecedents, which is important in solving data fusion problems [6]. In the present work we develop conditioning in a more general algebraic context which includes Boolean algebras and Zadeh's fuzzy set theory. Our structure, roughly speaking, has the properties of a Boolean algebra except for the law of excluded middle. Using functional image extension, the principle used to define the operations in [2]–[5], we define disjunction, conjunction, and complement operations on the conditional objects. Unlike the probability case, closure problems arise. They are handled by extending the set of conditional objects. We define a semantic evaluation on the extended set which, when applied to Boolean algebras, yields the familiar conditional probability evaluations and which associates to each conditional fuzzy set a fuzzy set membership function. L. Zadeh [10], E. Hisdall [7], and H. Nguyen [9] have also studied conditional fuzzy set membership functions.

### 1. Basic Syntactic Properties

In this section we define conditioning for semi-Boolean algebras, and we define algebraic operations on the set of conditional events. We obtain formulas for these operations when we restrict attention to Boolean algebras or Zadeh's fuzzy subsets.

**Definition 1.** A *semi-Boolean algebra* is a complete bounded distributive lattice,  $(S, \leq, \emptyset, \Omega)$ , with an involutive operator,  $()'$ , such that:

1. De Morgan's laws hold.
2.  $\emptyset' = \Omega$
3.  $\cdot$  is infinitely distributive over  $\vee$

Here  $\emptyset$  denotes the zero element of the lattice, and  $\Omega$  denotes the unit element. For  $a, b \in S$ ,  $a \cdot b = \sup\{x \in S \mid x \leq a, \text{ and } x \leq b\}$  and  $a \vee b = \inf\{x \in S \mid x \geq a \text{ and } x \geq b\}$ .

Semi-Boolean algebras differ from Boolean algebras, [8], in that we do not require:

1.  $x' \cdot x = \emptyset$ , or
2.  $x' \vee x = \Omega$ .

Note that Zadeh's fuzzy sets and Boolean algebras are semi-Boolean algebras. Furthermore, observe that the inequality is compatible with the operations. Namely:  $a \leq b \iff a = a \cdot b \iff b = a \vee b$

**Definition 2.** Let  $S$  be a semi-Boolean algebra, and let  $a, b \in S$ . The *conditional object*  $(a|b)$  is defined by

$$(a|b) \stackrel{\text{def}}{=} \{x \in S \mid x \cdot b = a \cdot b\}.$$

Furthermore:

$$(S|S) \stackrel{\text{def}}{=} \{(a|b) \mid a, b \in S\} \subset \mathcal{P}(S).$$

We use the following notation. Let  $a, b \in S$ . Recall, [8], that the relative pseudo-complement of  $b$  with respect to  $a$ , denoted here by  $b \triangleright a$ , is  $\sup\{x \in S \mid x \cdot b \leq a\}$ . One readily shows that  $b \triangleright (a \cdot b) = \sup\{x \in S \mid x \cdot b = a \cdot b\}$ . Additionally, we single out closed intervals of events:  $[a, b] \stackrel{\text{def}}{=} \{x \in S \mid a \leq x \leq b\}$ .

Let  $X$  and  $Y$  be sets and let  $f : X \rightarrow Y$  be an arbitrary function. Let  $P(X)$  denote the power class of  $X$ . Using the functional image extension, we define a map  $\hat{f} : P(X) \rightarrow P(Y)$ . Precisely, if  $A \in P(X)$ , then  $\hat{f}(A) \stackrel{\text{def}}{=} \{f(x) \mid x \in A\}$ . Among extensions,  $g : P(X) \rightarrow P(Y)$ , of  $f$  satisfying: if  $A \subset B$  then  $g(A) \subset g(B)$ ,  $\hat{f}$  is minimal, i.e. for all  $A \in P(X)$   $\hat{f}(A) \subset g(A)$ .

**Theorem 1.** Let  $S$  be a semi-Boolean algebra and let  $a, b \in S$ . Then:

$$(a|b) = (a \cdot b|b) = [a \cdot b, b \triangleright (a \cdot b)]. \quad (1)$$

Consequently, without loss of generality, one may assume that, in the expression  $(a|b)$ ,  $a \leq b$ . Furthermore,

$$(a|b) = (c|d) \iff a \cdot b = c \cdot d \text{ and } b \triangleright (a \cdot b) = d \triangleright (c \cdot d),$$

and

$$(a|\Omega) = [a \cdot \Omega, \Omega \triangleright (a \cdot \Omega)] = \{a\}.$$

**Theorem 2.** Let  $S$  be a semi-Boolean algebra, and suppose  $f : S^n \rightarrow S$  is a surjective function. For  $j = 1, \dots, n$  and  $a_j, b_j \in S$  define:

$$\alpha \stackrel{\text{def}}{=} f(a_1 \cdot b_1, \dots, a_n \cdot b_n),$$

and

$$\beta \stackrel{\text{def}}{=} f(b_1 \triangleright (a_1 \cdot b_1), \dots, b_n \triangleright (a_n \cdot b_n)).$$

If  $f$  is nondecreasing with respect to  $\leq$ , then:

$$f((a_1|b_1), \dots, (a_n|b_n)) = [\alpha, \beta], \quad (2)$$

and if  $f$  is nonincreasing with respect to  $\leq$ , then:

$$f((a_1|b_1), \dots, (a_n|b_n)) = [\beta, \alpha]. \quad (3)$$

**Corollary 1.** Suppose  $S$  is a semi-Boolean algebra. Then, using functional image extension,  $\cdot$ ,  $\vee$ , and  $()'$  extend to  $\mathcal{P}(S)^2$ . For all  $a, b, c, d \in S$ :

$$(a|b) \vee (c|d) = [ab \vee cd, b \triangleright (a \cdot b) \vee d \triangleright (c \cdot d)]. \quad (4)$$

$$(a|b) \cdot (c|d) = [ab \cdot cd, (b \triangleright (a \cdot b)) \cdot (d \triangleright (c \cdot d))]. \quad (5)$$

If  $()'$  is nonincreasing, then:

$$(a|b)' = [(b \triangleright (a \cdot b))', (a \cdot b)']. \quad (6)$$

In general, the operations are not closed i.e., given  $a, b, c, d \in S$  there may not exist  $\alpha$  and  $\gamma$  such that:

$$\begin{aligned} a \cdot b \vee c \cdot d &\leq \alpha; \quad a \cdot b \cdot c \cdot d \leq \gamma, \\ (b \triangleright (a \cdot b)) \vee (d \triangleright (c \cdot d)) &= \alpha \triangleright (a \cdot b \vee c \cdot d), \text{ or} \\ (b \triangleright (a \cdot b)) \cdot (d \triangleright (c \cdot d)) &= \gamma \triangleright (a \cdot b \cdot c \cdot d). \end{aligned}$$

However closure does hold when  $S$  is a Boolean algebra, see [5] and theorem 3 below. The closure of the operations and the identification of the results as conditional objects are easier to obtain in the present context than in that of [5].

**Theorem 3.** ([2], theorem 3.1, or [5], section 3.3) Let  $S$  be a Boolean algebra and let  $a, b, c, d \in S$ .

$$\begin{aligned} (a|b) \vee (c|d) &= (ab \vee cd | ab \vee cd \vee bd) \\ &= (ab \vee cd | ab \vee cd \vee a'bc'd); \end{aligned} \quad (7)$$

$$\begin{aligned} (a|b) \cdot (c|d) &= (abcd | a'b \vee c'd \vee bd) \\ &= (abcd | a'b \vee c'd \vee abcd); \end{aligned} \quad (8)$$

$$(a|b)' = (a'|b). \quad (9)$$

If  $S$  is the set of Zadeh's fuzzy subsets, with the operations  $\min$ ,  $\max$  and  $1 - ()$ ,  $S$  is a semi-Boolean algebra. See [1].  $(S|S)$  is not closed under the operations  $\cdot$  and  $()'$ , however it is closed under  $\vee$ . For fuzzy subsets  $a$  and  $b$  of a set  $X$ , we define the fuzzy subset  $\delta_{a,b}$  by

$$\delta_{a,b}(x) = \begin{cases} 1, & \text{if } a(x) = b(x) \text{ for } x \in X, \\ 0, & \text{otherwise.} \end{cases}$$

**Corollary 2.** Suppose  $S$  is Zadeh's fuzzy subset system. Then for all  $a, b, c, d \in S$ :

$$b \triangleright (a \cdot b) = a \cdot b \vee \delta_{ab,b}. \quad (10)$$

$$(a|b) \vee (c|d) = (ab \vee cd | ab \vee cd \vee \delta'_{ab,b} \cdot \delta'_{cd,d}). \quad (11)$$

$$(a|b) \cdot (c|d) = [abcd, (ab \vee \delta_{ab,b}) \cdot (cd \vee \delta_{cd,d})]. \quad (12)$$

$$\begin{aligned} (a|b)' &= [(ab \vee \delta_{ab,b})', (ab)'] \\ &= [(ab)' \cdot \delta'_{ab,b}, (ab)']. \end{aligned} \quad (13)$$

For each  $1 \leq j \leq n$  let  $a_j, b_j \in S$  and without loss of generality assume that  $a_j \leq b_j$ . The  $n$ -argument extensions for  $\cdot$  and  $\vee$  of corollary 2 are:

$$\begin{aligned} (a_1|b_1) \vee \dots \vee (a_n|b_n) \\ = (a_1 \vee \dots \vee a_n | a_1 \vee \dots \vee a_n \\ \vee (\delta_{a_1,b_1})' \cdot \dots \cdot (\delta_{a_n,b_n})'); \end{aligned} \quad (14)$$

and

$$\begin{aligned} (a_1|b_1) \cdot \dots \cdot (a_n|b_n) \\ = [a_1 \cdot \dots \cdot a_n, (a_1 \vee \delta_{a_1,b_1} \cdot \dots \cdot (a_n \vee \delta_{a_n,b_n}))]. \end{aligned} \quad (15)$$

The partial ordering on a semi-Boolean algebra  $S$  extends to the set of conditional events  $(S|S)$ . Define:

$$(a|b) \leq (c|d) \iff (a|b) = (a|b) \cdot (c|d).$$

Then

$$(a|b) \leq (c|d) \iff (c|d) = (c|d) \vee (a|b).$$

**Theorem 4.** Let  $S$  be the semi-Boolean algebra of fuzzy sets, and let  $a, b, c, d \in S$ .  $(a|b) \leq (c|d)$  if and only if:

$$a \cdot b \leq c \cdot d$$

and

$$b \triangleright (a \cdot b) \leq d \triangleright (c \cdot d).$$

The proofs of our results use the following lemmas.

**Lemma 1.** Let  $S$  be a semi-Boolean algebra.  $a, b, c \in S$ , and  $a \leq b$ .

$$(1.) \quad b \cdot (b \triangleright a) = a.$$

$$(2.) \quad a \leq b \triangleright a.$$

$$(3.) \quad b \leq (b \triangleright a) \triangleright a.$$

$$(4.) \quad c \triangleright b \leq c \triangleright a.$$

**Lemma 2.** The equation,  $\omega \triangleright a = b$ , has a solution if and only if  $(b \triangleright a) \triangleright a = b$ .

**Definition 3.** The semi-Boolean algebra  $S$  has the *surjective pseudoinverse* property if for all  $a, b \in S$   $\omega \triangleright a = b$ , has a solution, or equivalently,  $(b \triangleright a) \triangleright a = b$ .

**2. Conditional Event and Closed Interval Algebras and Semantic Evaluations**

We define the closed interval algebra,  $\mathcal{I}(S)$ , of a semi-Boolean algebra  $S$ . If  $S$  is a Boolean algebra then  $\mathcal{I}(S) = (S|S)$ , while if  $S$  is Zadeh's fuzzy set system,  $\mathcal{I}(S)$  is the closure of  $(S|S)$  under the above mentioned operations. For an arbitrary semi-Boolean algebra  $S$  and compatible t-norm, we define a semantic evaluation on  $\mathcal{I}(S)$ . For Boolean algebras this yields the usual conditional probability, and it associates to each closed interval of fuzzy sets a fuzzy set membership function. Finally, we compare the closed interval approximation and the best-upper approximation of composites of conditional fuzzy sets.

We single out certain subsets of  $\mathcal{P}(S)$  for an arbitrary semi-Boolean algebra  $S$ .

$$\mathcal{I}(S) \stackrel{\text{def}}{=} \{[a, c] \mid a \leq c \in S\};$$

$$\mathcal{J}(S) \stackrel{\text{def}}{=} \{[b, \Omega] \mid b \in S\};$$

$$\mathcal{K}(S) \stackrel{\text{def}}{=} \{\{d\} \mid d \in S\}.$$

$$\mathcal{M}(S) \stackrel{\text{def}}{=} \{(a|b) \cdot (c|d) \mid a, b, c, d, \in S\}.$$

$\mathcal{C}(S|S)$  is the finite closure of  $(S|S)$  under the operations  $\cdot$ ,  $\vee$ , and  $()'$ .

**Theorem 5.** Let  $S$  be a Boolean algebra.

1. For all  $a, c \in S$ , the equation  $x \triangleright a = c$  has a unique solution.
2.  $(S|S) = \mathcal{I}(S)$ .

**Theorem 6.** Let  $S$  be Zadeh's fuzzy set system.

1. The equation,  $x \triangleright a = c$ , has a solution if and only if:
  - a.  $a = c < \Omega$ ,
  - b.  $a < c = \Omega$ , or
  - c.  $a = c = \Omega$ .
- 2.

$$\begin{aligned} S &\subset (S|S) \\ &= \mathcal{J}(S) \cup \mathcal{K}(S) \\ &\subset \mathcal{I}(S) \\ &\subset \mathcal{P}(S). \end{aligned}$$

**Theorem 7.** Let  $S$  be Zadeh's fuzzy set system.

Then

$$\begin{aligned} \mathcal{C}(S|S) &= \mathcal{M}(S) \\ &= \mathcal{I}(S). \end{aligned}$$

With theorem 7 established, it is clear that semantic evaluations for any Boolean operations acting upon conditional events can be carried out, provided such evaluations are well defined upon all closed intervals of events. Let  $S$  be a semi-Boolean algebra, and let  $\phi$  be a compatible t-norm. For Boolean algebras we take  $\phi$  to be multiplication, and for Zadeh's fuzzy sets we take  $\phi$  to be min.

**Definition 4.** For real numbers  $0 \leq \lambda_1 \leq \lambda_2 \leq 1$ , define the *conditional number*  $(\lambda_1|\lambda_2) = \sup\{\lambda \mid 0 \leq \lambda \leq 1, \text{ and } \phi(\lambda, \lambda_2) = \lambda_1\}$ .

**Definition 5.** If  $\| \cdot \|$  is any model for  $S$ , and if  $a \leq c \in S$ , define:

$$\|[a, c]\| = (\|a\| \|c \triangleright a\|).$$

**Theorem 8.** Let  $S$  be a Boolean algebra,  $\| \cdot \| : S \rightarrow [0, 1]$  a finitely additive probability measure, and  $\phi : [0, 1]^2 \rightarrow [0, 1]$  multiplication. Then, assuming  $a \leq b$ ,

$$\begin{aligned} \|(a|b)\| &= \|[a, b \triangleright a]\| \\ &= \begin{cases} \|a\|/\|b\|, & \text{if } \|b\| > 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

**Theorem 9.** Let  $S$  be Zadeh's fuzzy set system, with  $\|a\|$  the fuzzy set membership function of  $a \in S$ , and  $\phi = \min$ .

1. For all  $0 \leq \lambda_1 \leq \lambda_2 \leq 1$ ,

$$(\lambda_1|\lambda_2) = \max(\lambda_1, \delta_{\lambda_1, \lambda_2}).$$

- 2.

$$\|[a, c]\| = \max(\|a\|, \delta_{\|a\|, \|c\|}).$$

- 3.

$$\|(a|b)\| = \max(\|a\|, \delta'_{\|a\|, \|b\|}).$$

Thus, theorem 9 shows that definition 5 is reasonable, as it reduces to the usual conditional probability evaluation for Boolean algebras. It provides for semantic evaluation of any Boolean combination of Zadeh's fuzzy set conditional events, or any operation yielding closed intervals. We compare the above technique with the optimal upper approximation introduced in [5] section 6.

**Definition 6.** Let  $S$  be a semi-Boolean algebra, and let  $A \subset S$ . The *optimal upper approximation* of  $A$  with respect to  $(S|S)$  is:

$$\text{cond}(A) = \bigcap \{(a|b) \mid A \subset (a|b), a, b \in S\}.$$

For  $A \subset S$  let:

$$\cdot(A) \stackrel{\text{def}}{=} \cdot\{x \mid x \in A\}; \quad \vee(A) \stackrel{\text{def}}{=} \vee\{x \mid x \in A\}.$$

**Theorem 10.** Suppose  $S$  is a semi-Boolean algebra.

- (i.)  $\text{cond}(A) = \text{cond}(\cdot(A), \vee(A))$
- (ii.)  $\text{cond}([a, b]) = [a, (b \triangleright a) \triangleright a] = (a|b \triangleright a)$

**Theorem 11.**

1. If  $S$  possesses the surjective pseudoinverse property, then for all  $a \leq b \in S$ ,

$$\begin{aligned} \text{cond}[a, b] &= [a, b] \\ &= (a|b \triangleright a). \end{aligned}$$

2. If  $S$  is an arbitrary semi-Boolean algebra, then:

$$||[a, b]|| = ||\text{cond}[a, b]||.$$

**Theorem 12.** Suppose  $S$  is Zadeh's fuzzy subset system. Assume, without loss of generality, that  $a \leq b \in S$  and  $a_j \leq b_j \in S$  for  $1 \leq j \leq n$ . Identify  $||\delta_{a,b}||$  with  $\delta_{a,b}$ . Furthermore, let  $T_n = \{j \mid a_j < b_j\}$  and  $T'_n = \{j \mid a_j = b_j\}$ . Then:

$$\text{cond}((a|b)') = (a' \cdot \delta'_{a,b} | \delta'_{a,b} \vee \delta_{a,n}). \quad (16)$$

$$\text{cond}((a_1|b_1) \cdots (a_n|b_n)) = (\alpha_2 | \beta_2 \triangleright \alpha_2). \quad (17)$$

where

$$\begin{aligned} \alpha_2 &= a_1 \cdots a_n; \\ \beta_2 &= (a_1 \vee \delta_{a_1,b_1}) \cdots (a_n \vee \delta_{a_n,b_n}) \\ &= \{a_j \mid j \in T_n\}. \\ \beta_0 &\stackrel{\text{def}}{=} \{a_j \mid j \in T'_n\} \\ \beta_2 \triangleright \alpha_2 &= a_1 \cdots a_n \cdot \delta_{(\beta_2 \leq \beta_0)}. \end{aligned}$$

**Corollary 3.** The norm of combinations of conditional fuzzy sets are:

$$\begin{aligned} ||(a|b)'|| &= ||\text{cond}((a|b)')|| \\ &= \max(\delta_{a,n}, \delta_{a,b}). \end{aligned} \quad (18)$$

$$\begin{aligned} ||(a_1|b_1) \vee \cdots \vee (a_n|b_n)|| \\ &= \max(||\alpha_1||, \delta'_{a_1,b_1}, \dots, \delta'_{a_n,b_n}), \end{aligned} \quad (19)$$

where

$$\alpha_1 = a_1 \vee \cdots \vee a_n.$$

$$\begin{aligned} ||((a_1|b_1) \cdots (a_n|b_n))|| \\ &= ||\text{cond}((a_1|b_1) \cdots (a_n|b_n))|| \\ &= \max(||\alpha_2||, \delta_{(\beta_2 \leq \beta_0)}, \delta'_{\beta_0, ||\alpha_2||}, \delta'_{(\beta_2 \leq \beta_0)}) \end{aligned} \quad (20)$$

### Concluding Remarks

In summary it should be emphasized that for logical systems, which can be characterized as semi-Boolean algebras, conditional objects can be identified as totally ordered chains, or equivalently, closed intervals.

of unconditional events with a relative pseudocomplement relation connecting the endpoints. This allows for ease of computing the functional image extension of any conditional object, and yields almost the same algebraic properties as for the underlying unconditional objects. Unlike the Boolean case, Zadeh's system produces closure problems for the extension of  $\cdot$  and  $()'$ . This difficulty is addressed by utilizing the optimal upper approximation technique.

Finally, one could question the present development of conditioning, which extends the syntactic approach to conditioning in probability previously developed by one of the authors, as opposed to the much simpler material implication. If the basic analogue between conditioning in probability and that for other logical systems is to be kept, one cannot utilize material implication, and one is forced to go in the more syntactic direction as provided here. Future efforts will address this and related issues.

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